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Directed percolation in two and three dimensions: II. Direction dependence of the wetting velocity

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Abstract. We study the behaviour of the average velocity of fluid flow in a random network made of oriented diodes and resistors, as a function of the direction of the fluid flow and the concentration of resistors. It is shown that there is more than one critical value of the concentration across which there is a qualitative change in the behaviour of the wetting velocity. Also, it is shown that the directed percolation problem is related to the problem of determining the direction-dependent wetting velocity in the undirected percolation problem.

1. Introduction

In the preceding paper (Dhar 1982) we have studied the percolation properties of a random network of oriented diodes and resistors. These circuit elements are arranged to form the bonds of a regular lattice (say, a square lattice in two dimensions, or a simple cubic lattice in three dimensions). Each bond of the lattice can be a diode or a resistor, independent of other bonds, with probability q and p respectively ($p+q=1$). The direction of the diodes is assumed to be in the direction of increasing x, y or z coordinates and is not a random variable. It was shown that there exists a critical percolation probability p_c such that, if the concentration of resistors p is greater than p_c, a point source of fluid will wet all sites of the lattice with probability 1.

The percolation problem for the diode–resistor percolation is seen to be trivial if $p \geqslant p_c$. This, of course, does not imply that the diode–resistor network has no interesting properties in this range of p. In this paper we study the direction dependence of the wetting velocity in these networks. We show that, in general, there is more than one critical value of the resistor concentration p across which the wetting velocity undergoes a qualitative change. We also discuss the behaviour of the wetting velocity in the undirected percolation problem and establish its relationship to the directed percolation problem.

The wetting velocity is defined as follows. We start with a configuration of diodes and resistors on the square lattice with all sites dry. At time $t=0$, a point source of fluid is introduced at the origin. We assume that the fluid takes one unit of time to traverse a bond in an allowed direction, and cannot flow in the disallowed direction (the corresponding traversal time is infinite). As time passes, more and more of the sites of the lattice are wetted, and the boundary of the wetted cluster moves outward. We define the wetting velocity $v(\hat{\phi}, p)$ as the average velocity of this boundary in the direction $\hat{\phi}$.

To be more precise, let $T(\boldsymbol{R})$ be the time when the fluid first reaches the site $\boldsymbol{R}$. It can be shown that $T(n\boldsymbol{R})/n$ converges in probability to some value $T_{av}(\boldsymbol{R})$ as n tends to infinity. We define $(|\boldsymbol{R}|)/T_{av}(R)$ as the average velocity in the direction $\hat{\phi} = \boldsymbol{R}/|\boldsymbol{R}|$.

The problem of the determination of the wetting velocity is known as the first-passage percolation problem in the literature. Hammersley and Welsh (1965) defined the problem for arbitrary graphs with a more general distribution of traversal times. They proved the existence of the large time limit used in the definition of the wetting velocity for a wide class of distributions of traversal times, and also obtained some lower-bound estimates for the wetting velocity in the case of undirected percolation on a square lattice. Two recent reviews of the first-passage percolation theory are by Smythe and Wierman (1978) and Hammersley and Welsh (1980). Janson (1981) has obtained upper-bound estimates for the wetting velocity in the undirected percolation problem. Directional effects in percolation problems have attracted much attention lately. Domany and Kinzel (1981) have studied the direction dependence of the correlation length near criticality in an exactly soluble model. In this paper we discuss the non-analytic behaviour of the wetting velocity as a function of concentration, and its directional dependence. A related problem—describing the shape of the infected cluster in an epidemic model—has been discussed recently by Durrett and Liggett (1981).

2. Wetting velocity in the diode–resistor percolation

In this section we discuss the direction dependence of the wetting velocity in diode-resistor percolation. For simplicity, we shall explicitly consider only the two-dimensional square lattice. The generalisation to three dimensions is quite straightforward and is mentioned briefly at the end of the section.

The direction $\hat{\phi}$ can be specified by the angle ϕ the vector $\hat{\phi}$ makes with the direction $x = y > 0$. It is convenient to define the distance between two points (x_1, y_1) and (x_2, y_2) as $|x_2 - x_1| + |y_2 - y_1|$, and not by the Euclidean metric. It follows immediately that the wetting velocity $v(\phi, p)$ cannot exceed 1.

As upward and rightward flow is always allowed,

$$v(\phi, p) = 1 \qquad \text{for } \phi \leq \pi/4 \quad \text{for all } p. \tag{1}$$

If $p < p_c$, wetting occurs only within a wedge of half-wedge angle $\theta(p)$. This implies that

$$v(\phi, p) = 0 \qquad \text{for } \phi > \theta(p) \quad p < p_c. \tag{2}$$

Consider a wetting strategy in which the next step is chosen to be leftward, if allowed, otherwise upward. The average direction of motion using this strategy is easily seen to be $\phi = \cot^{-1}(1-2p)$, and the wetting velocity is clearly 1. The wetting velocity will be 1 for all $\phi < \cot^{-1}(1-2p)$, as (say in the second quadrant) we can go directly upward once the desired x coordinate has been reached. Thus equation (1) may be improved to

$$v(\phi, p) = 1 \qquad \text{if } \phi < \cot^{-1}(1-2p). \tag{3}$$

Since wetting in a direction $\phi > \cot^{-1}(1-2p)$ must involve some backflow, it seems reasonable to conjecture that $v(\phi, p)$ is strictly less than 1 if $p < p_c$ and $\theta(p) > \phi > \cot^{-1}(1-2p)$. We can derive lower-bound estimates to $v(\phi, p)$ which show that $v(\phi, p)$

is strictly greater than zero in this regime. Consider, for example, the strategy $k=1$ (Dhar 1982). It is easy to see that, using the route determined by this strategy, a fluid particle moves in the average direction $\phi = \tan^{-1}[(q-p^2)/(q^2-p)]$ with an average velocity $(q-p^2)/(q+p^2)$. The true wetting velocity must be larger than this. This gives

$$v(\tan^{-1}[(q-p^2)/(q^2-p)], p) \geqslant (q-p^2)/(q+p^2). \tag{4}$$

If we re-define the wetting path, eliminating the unnecessary re-traversals implicit in the strategy $k=1$, this bound can be increased to

$$v(\tan^{-1}[(q-p^2)/(q^2-p)], p) \geqslant (q-p^2)(1-pq)/(p^3+q^3+pq-p^2q^2). \tag{5}$$

In order to go in a direction ϕ, $\cot^{-1}(1-2p) < |\phi| < \cot^{-1}[(q^2-p)/(q-p^2)]$, one may first go a distance R_1 using strategy $k=0$, and then go a distance $R-R_1$ in the direction $\cot^{-1}(1-2p)$ with velocity 1, the distance R_1 being chosen such that the total displacement is in the direction ϕ. The average velocity thus obtained lies between 1 and $v(\tan^{-1}[(q-p^2)/(q^2-p)], p)$, and is a lower bound to the velocity $v(\phi, p)$.

The $k=1$ strategy cannot be used to estimate the wetting velocity in directions $\phi > \tan^{-1}[(q-p^2)/(q^2-p)]$. For obtaining lower-bound estimates to $v(\phi, p)$ we need to use $k=2$, $k=3$ strategies, etc which involve more and more backflows. In the limiting case $\phi = \theta(p)$ we need to consider arbitrarily large backflows, as any strategy ignoring backflows of order larger than a given finite integer k will not be able to wet in the direction $\phi = \theta(p)$. The determination of the behaviour of $v(\phi, p)$ as a function of ϕ in the neighbourhood of $\phi = \theta(p)$ is an interesting and difficult problem requiring further investigation.

A plausible conjecture is that $v(\phi, p)$ decreases monotonically from 1, and tends to a finite value $v_{\text{disc}}(p)$, as ϕ is increased from $\cot^{-1}(1-2p)$ to $\theta(p)$. At $\phi = \theta(p)$ the function is discontinuous, and is zero for all larger values of ϕ. The size of the discontinuity $v_{\text{disc}}(p)$ is 1 at $p=0$ and decreases monotonically as p is increased to p_c. For p just below p_c we expect that $v_{\text{disc}}(p)$ varies as a power of $p_c - p$ (this critical exponent may be zero).

The dependence of $v(\phi, p)$ is not monotonic in ϕ. To demonstrate this result, we first determine an upper bound on $v(\phi, p)$.

Consider a point $P \equiv (-m, -n)$ in the third quadrant far away from the origin. Let $N(L, m, n, s)$ be the number of self-avoiding walks from the origin to P of length L involving s rightward steps. Clearly such a random walk involves $m+s$ steps leftward, $(L-m-n-2s)/2$ steps upward, and $(L-m+n-2s)/2$ steps downward. Clearly, $N(L, m, n, s)$ is less than the number of all possible paths (self-avoiding or not) having length L, ending at $(-m, -n)$ and having s rightward steps. Thus

$$N(L, m, n, s) \leqslant L!/\{(m+s)!\,s!\,[(L-m-n-2s)/2]!\,[(L-m+n-2s)/2]!\}. \tag{6}$$

The total number of paths $N(L, m, n)$ is obtained by summing $N(L, m, n, s)$ over all allowed values of s, $(L-m-n)/2 \geqslant s \geqslant 0$:

$$N(L, m, n) = \sum_s N(L, m, n, s). \tag{7}$$

The probability that a particular self-avoiding path from the origin to P of length L is allowed is $p^{(L+m+n)/2}$. The probability that at least one of the $N(L, m, n)$ paths is allowed is less than or equal to $N(L, m, n)\ p^{(L+m+n)/2}$. Finally, we sum all paths of length less than or equal to L to get an upper bound on the probability that at least one of them is allowed (i.e. the wetting time T up to the point P is less than or equal to L). We get

$$\Pr(T \leqslant L) \leqslant \sum_{l=m+n}^{L} p^{(l+m+n)/2} N(l, m, n). \tag{8}$$

For small values of n/m there exists a value $V_b < 1$ such that for all $L/m < V_b^{-1}$, the right-hand side can be made arbitrarily small in the limit $m, n, L \to \infty$ with n/m and L/m held constant. This provides us with an upper bound on the wetting velocity

$$v(\phi, p) \leqslant [1 + |\cot(\phi - \pi/4)|]/x \tag{9}$$

where x is the largest positive real number such that

$$\lim_{m\to\infty} m^{-1} \log N(xm, m, m|\cot(\phi - \pi/4)|) \leqslant -[1 + x + \cot(\phi - \pi/4)](\log p)/2. \tag{10}$$

Numerical calculation shows that for $p = 0.7, 0.8, 0.9$, $v(3\pi/4, p)$ is less than 0.9181, 0.9543, 0.9816 respectively. These upper bounds on $v(\phi, p)$ can obviously be improved by using sharper estimates of $N(L, m, n, s)$, etc. The present bound is, however, quite sufficient for our purpose here. It shows that $v(\phi, p)$ is strictly less than 1 in the neighbourhood of $\phi = 3\pi/4$ for all values of $p < 1$. This is intuitively reasonable, as in the direction $\phi = 3\pi/4$ all paths with wetting velocity 1 are essentially one-dimensional. The formal proof is somewhat long because it is necessary to show that the probability of the existence of allowed paths of length $m + Am^{\alpha}$ to the site $(-m, 0)$ tends to zero as m tends to infinity for all $\alpha < 1$.

Surprisingly, the above result is not true for any other value of ϕ. There exists a value $p_{c2}(\phi)$ such that for all $p > p_{c2}(\phi)$ the wetting velocity $v(\phi, p)$ is exactly 1. For $\phi < 3\pi/4$ this result follows from equation (3). For $\phi > 3\pi/4$ this may be seen as follows. We know from duality arguments (Dhar *et al* 1981) that the critical probability for diode–insulator percolation (DIP) on a square lattice is $1 - p_c$. Let us consider paths in a diode–resistor configuration on a square lattice which involve only leftward or downward steps. With this additional constraint, the resistors act as diodes (allowing leftward or downward flow) and the diodes as insulators (allowing no flow). If $p > 1 - p_c$, there are infinite connected paths in this problem, and we can go arbitrarily far using only leftward or downward paths, without any re-traversals. The directions in which we can go are bounded by $\pi \pm \theta^{DIP}(p)$, where $\theta^{DIP}(p)$ is the half-wedge angle of the infinitely wetted cluster in DIP. Clearly the wetting velocity along these paths is 1. Also, by duality, $\theta^{DIP}(p)$ is $\pi/2 - \theta(1-p)$. Hence we get

$$v(\phi, p) = 1 \qquad \text{for } \phi > \pi/2 + \theta(1-p) \quad \text{and} \quad p > 1 - p_c. \tag{11}$$

These results are summarised in figure 1. For $p < p_c$, $v(\phi, p)$ is 1 for $\phi < \cot^{-1}(1-2p)$. It is non-unity and positive for $\cot^{-1}(1-2p) < \phi \leqslant \theta(p)$, and zero otherwise. There is a discontinuity in $v(\phi, p)$ at $\theta(p)$ (curve A). If $1 - p_c > p > p_c$, $v(\phi, p)$ is equal to 1 up to some value of ϕ, and strictly less than 1 for all higher values (curve C). For $p > 1 - p_c$, $v(\phi, p)$ is 1 except in an interval which includes the point $\phi = 3\pi/4$ (curve E). The size of this interval shrinks as p tends to 1. The curves B and D describe the expected behaviour of the wetting velocity at $p = p_c$ and $p = 1 - p_c$ respectively.

The behaviour of the wetting velocity in three dimensions is qualitatively the same. Of course, in three dimensions there is no simple relation between p_c^{DIP} and p_c^{DRP}.

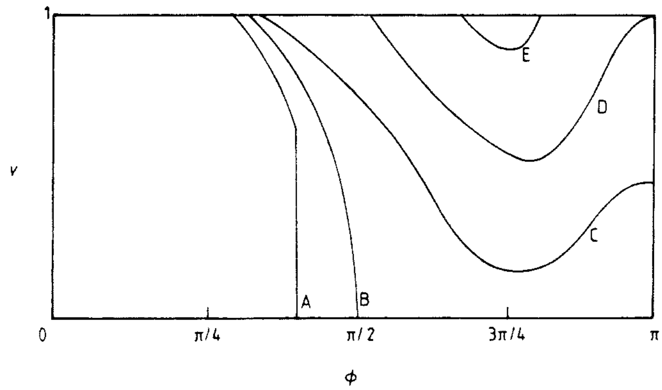

Figure 1. The qualitative behaviour of the wetting velocity $v(\phi, p)$ as a function of the direction ϕ for various values of p. The curves A, B, C, D and E correspond respectively to the cases $p < p_c$, $p = p_c$, $p_c < p < 1 - p_c$, $p = 1 - p_c$ and $p > 1 - p_c$.

If p exceeds p_c^{DIP} there are directions in which the wetting velocity is exactly 1. Also, for all $p < 1$, the wetting velocity along the negative coordinate axes is always less than 1.

3. Direction dependence of wetting velocity in undirected percolation

It is interesting to note that the above arguments may be generalised to the case of undirected percolation as well. In this case, there is always a non-zero probability that a source at origin will wet only a finite number of sites, or that a test site far away will be left dry even if the source belongs to an infinite cluster. In defining wetting velocity, it is reasonable to ignore such configurations. We may assume that the source belongs to the infinite cluster, and define wetting velocity as the velocity of the outer boundary of the wetted cluster (we ignore holes). With this convention, in this case too, if $p > p_c^{DIP}$, there are directions in which the wetting velocity is exactly 1. Also, along the coordinate axes the wetting velocity is less than 1 for all $p < 1$. Hence the *wetting velocity is not isotropic in the case of undirected percolation.*

In three dimensions, if p is slightly greater than p_c^{DIP}, the directions with wetting velocity equal to 1 lie inside eight narrow cones with axes near the directions $\pm x = \pm y = \pm z > 0$. As p is increased, these cones subtend larger solid angles, and beyond a certain value of p (this critical value must be less than or equal to p_c^{DIP} on a square lattice) they merge with each other and we have instead six disjoint cones with axes in the direction of the coordinate axes such that the wetting velocity inside these cones is strictly less than 1.

We have thus established a correspondence between the wetting velocity in the undirected percolation and the problem of directed percolation. For example, the dependence of the wedge angle on the probability of bonds in the directed percolation may be deduced from the calculation of directions with wetting velocity 1 in the corresponding undirected percolation problem. In the directions where the wetting velocity is 1, the difference between the expected wetting time to a point and its distance from the origin is a measure of the average separation between 'highways' in the problem. (The terminology is due to Hammersley and Welsh (1965).) Clearly, the problem of the determination of the direction dependence of the wetting velocity is very interesting, and its elucidation would improve our understanding of percolation transitions considerably. This looks like an interesting problem for further study.

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